

# Scaling of the Lyapunov exponent in type-III intermittent chaos

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The scaling behaviour of the Lyapunov exponent near the transition to chaos via type-III intermittency is determined for a generic map. A critical exponent  $\beta$  expressing the scaling of the Lyapunov exponent as a function of both, the reinjection probability and the nonlinearity of the map is calculated. It is found that the critical exponent varies on the interval  $0 < \beta < 1$ . This contrasts with earlier predictions for the scaling behaviour of the Lyapunov exponent in type-III intermittency.

PACS numbers: 05.45.Ac, 89.75.Da, 64.60.Fr

## 1. Introduction

Intermittency is a common scenario for the transition to chaos in nonlinear dynamical systems. Intermittent chaos is characterised by the display of long sequences of periodiclike behaviour, called the laminar phases, interrupted by comparatively short chaotic bursts. The phenomenon has been extensively studied since the original work of Pomeau and Manneville [1] classifying type-I, -II, and -III instabilities when the Floquet multipliers of the local Poincaré map associated to the system crosses the unit circle. Type-I intermittency occurs by a tangent bifurcation when the Floquet's multiplier for the Poincaré map crosses the circle of unitary norm in the complex plane through +1; type-II intermittency is due to a Hopf's bifurcation which appears as two complex eigenvalues of the Floquet's matrix cross the unitary circle off the real axis; and type-III intermittency is associated to an inverse period doubling bifurcation whose Floquet's multiplier is  $-1$  [2]. Although other mechanisms may occur leading to intermittency, these three cases are the most simple and the most frequently encountered in low-dimensional systems.

Many experimental evidences for these types of intermittency have appeared in the literature. In particular, type-III intermittency has been found in lasers [3], electronic nonlinear devices [4, 5, 6], biological tissues [7], and epilepsy [8]. The statistical signature of intermittency is usually given by the scaling relations describing the dependence of the average length (denoted by  $\langle l \rangle$ ) of the laminar phases with a control parameter (denoted by  $\epsilon$ ) that measures the distance from the bifurcation point. Chaotic dynamics is characterised by the positive sign of the largest Lyapunov exponent (denoted by  $\lambda$ ), although this quantity is in general more difficult to measure from

experimental data than statistical variables such as the average laminar length. Pomeau and Manneville [1] assumed a uniform reinjection probability into the laminar phase and calculated the scaling behaviour of both the average laminar length and the Lyapunov exponent for those three types of intermittent chaos. Specifically, for type-III intermittency Pomeau and Manneville predicted the relations  $\langle l \rangle \propto \epsilon^{-1}$ , and  $\lambda \propto \epsilon^{1/2}$ , both when  $\epsilon \rightarrow 0$ .

However, theoretical and numerical studies [9, 10, 11, 12], as well as some recent experiments [6, 7, 13, 14, 15, 16], have shown deviations from the Pomeau-Manneville's prediction for the scaling behaviour of the average laminar length for type-III intermittency; they found  $\langle l \rangle \propto \epsilon^{-\nu}$ , with  $1/2 < \nu < 1$ . These discrepancies are attributed to the presence of mechanisms leading to nonuniform probability reinjections. On the other hand, the scaling behaviour of the Lyapunov exponent in type-III intermittency has rarely been explored.

The aim of this paper is to clarify the effects of both the reinjection probability and the nonlinearity on the scaling properties of the Lyapunov exponent at the transition to chaos via type-III intermittency by using a simple one-dimensional map. We show that the behaviour of the Lyapunov exponent near this transition exhibits appreciable deviations from the prediction of Pomeau and Manneville. We find that  $\lambda \propto \epsilon^\beta$ , for  $\epsilon \rightarrow 0$ , where the critical exponent  $\beta$  varies continuously in the interval  $0 < \beta < 1$  as a function of the map parameters.

## 2. Scaling Behaviour of the Lyapunov Exponent

The return map is a fundamental information that can be obtained experimentally in a chaotic system and inevitable contains some reinjection process for type-III intermittency. Therefore, a reinjection mechanism should

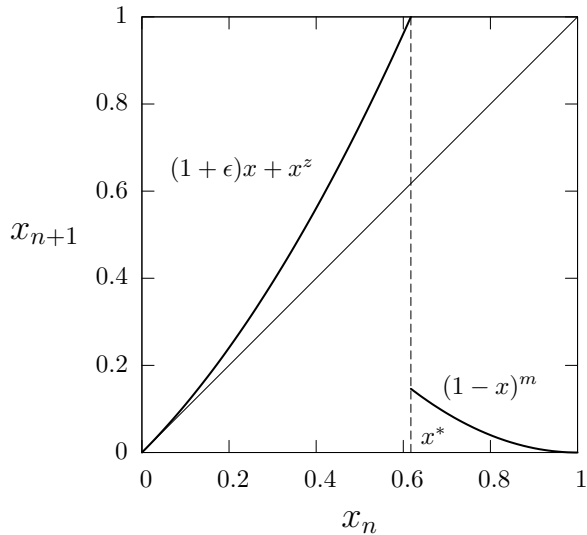


FIG. 1: Generic map Eq. (1) for producing type-III intermittency, with  $z = 2$ ,  $m = 2$  and  $\epsilon = 0.002$ .

be taken into account in the analysis of characteristic quantities for intermittency such as the mean laminar length or the Lyapunov exponent. We consider the following one-dimensional map [10]

$$x_{n+1} = f(x_n) = \begin{cases} (1 + \epsilon)x_n + x_n^z, & \text{if } x_n < x^* \\ (1 - x_n)^m, & \text{if } x_n > x^* \end{cases} \quad (1)$$

where  $x_n \in [0, 1]$ ,  $z \geq 2$ ,  $m \geq 1$  and  $x^*$  is the solution of

$$(1 + \epsilon)x^* + x^{*z} = 1. \quad (2)$$

The parameter  $\epsilon$  measures the distance from the bifurcation point. For  $\epsilon > 0$ , the map Eq. (1) generates type-III intermittency near the origin which becomes an unstable fixed point. The parameter  $m$  controls the reinjection probability into the laminar region of the map, with  $m = 1$  corresponding to uniform reinjection. The reinjection probability becomes more densely localised near the origin as  $m$  increases. Figure 1 shows the map Eq. (1).

The Lyapunov exponent is calculated as

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log |f'(x_n)|. \quad (3)$$

We have used Eq. (3) with  $N = 10^6$  for each set of parameter values  $z$ ,  $m$  and  $\epsilon$ , averaging over  $10^3$  random initial conditions  $x_0$  uniformly distributed on the interval  $(0, 1)$ , after discarding  $4 \times 10^3$  iterates as transients. Figure 2 shows the Lyapunov exponent for the map Eq. (1) as a function of the parameter  $\epsilon$ , for fixed  $z$  and several values of  $m$ , and for a fixed value of  $m$  and different values of the nonlinearity exponent  $z$ .

The onset of type-III intermittency is associated to a direct transition from a fixed point to a chaotic one-band

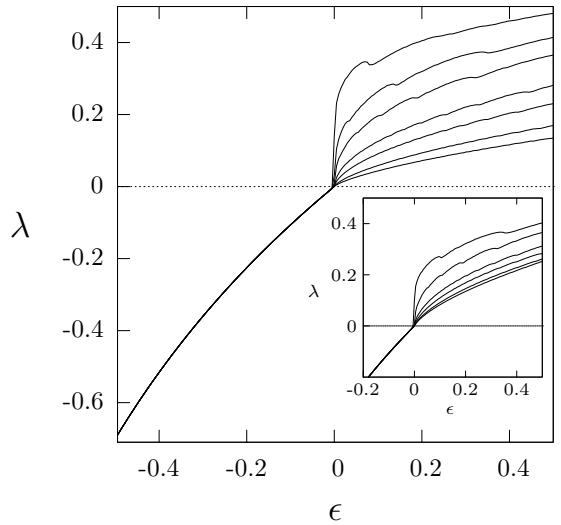


FIG. 2: Lyapunov exponent  $\lambda$  as a function of  $\epsilon$  for different values of  $m$  and with fixed  $z = 2$ ; from top to bottom  $m = 1, 1.5, 2, 3, 4, 6$  and  $8$ . Insert:  $\lambda$  as a function of  $\epsilon$ , with fixed  $m = 2$ , for different values of  $z$ ; from top to bottom  $z = 2, 3, 4, 5, 6$  and  $8$ .

attractor. The map Eq. (1) exhibits robust chaos for  $\epsilon > 0$ . A chaotic attractor is said to be robust if, for its parameter values, there exist a neighbourhood in the parameter space with absence of periodic windows (i.e. with  $\lambda > 0$ ) and the chaotic attractor is unique [17]. Robustness is an important property in applications that require reliable operation under chaos, in the sense that the chaotic behaviour cannot be destroyed by arbitrarily small perturbations of the system parameters. It should be noted that robust chaos not associated to type-III intermittency has also been discovered in smooth, continuous one-dimensional maps [18].

The transition to chaos via type-III intermittency is manifested by a discontinuity of the derivative of the Lyapunov exponent at the bifurcation value  $\epsilon = 0$ . This discontinuity is due to the sudden loss of stability of the fixed point associated to the inverse period doubling bifurcation that occurs at the onset of type-III intermittency. The Lyapunov exponent can be regarded as an order parameter that characterises the transition to chaos via type-III intermittency. This transition can be very abrupt as seen in 2. Thus, the behaviour of the Lyapunov exponent for  $\epsilon \rightarrow 0$  can be described by a scaling relation

$$\lambda \sim \epsilon^{\beta(z, m)}, \quad (4)$$

where  $\beta$  is a critical exponent expressing the order of the transition and that depends on the parameters  $z$  and  $m$ . Figure 3(a) shows a log-log plot of the Lyapunov exponent vs.  $\epsilon$  for a fixed value of  $z$  and several values of the reinjection parameter  $m$ , while Fig. 3(b) shows the same plot with a fixed value of  $m$  and for different values of the nonlinearity  $z$ . Thus, relation Eq. (4) is satisfied independently for each parameter  $z$  and  $m$ . The critical

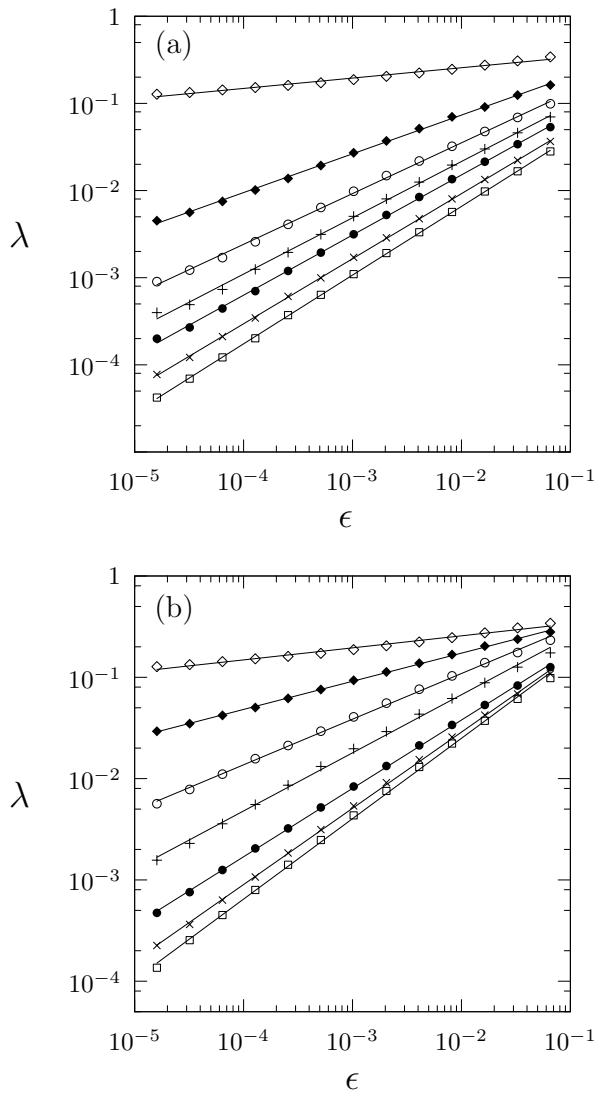


FIG. 3: (a) Log-log plot of the Lyapunov exponent  $\lambda$  vs.  $\epsilon$  with fixed  $z = 2$  for different values of  $m$ ; from top to bottom  $m = 1, 2, 3, 4, 5, 7$  and  $9$ . (b) Log-Log plot of  $\lambda$  vs.  $\epsilon$  with fixed  $m = 1$  for different values of  $z$ ; from top to bottom  $z = 2, 2.4, 3, 4, 6, 8$  and  $10$ .

exponent  $\beta$  as a function of  $z$  and as a function of  $m$  can be calculated from the slopes of each curve in Fig. 3(a) and 3(b), respectively.

Figure 4 shows the resulting graph of the critical exponent  $\beta$  as a function of both  $z$  and  $m$ . The values of  $\beta$  are distributed continuously on the interval  $(0, 1)$ . We find the value  $\beta = 0.495 \pm 0.005$  (typical error) for the parameter values  $z = 3$  and  $m = 1$ , for which Eq. (1) corresponds to the tangent map originally associated to type-III intermittency and having uniform reinjection probability, as assumed by Pomeau and Manneville [1, 2]. However, for other values of  $z$  and  $m$  there is a manifested deviation from the Pomeau-Manneville's prediction  $\beta = 1/2$ .

The calculated range for the critical exponent includes the particular value of  $\beta \approx 0.79$  obtained for a perturbed logistic map showing type-III intermittency [19].

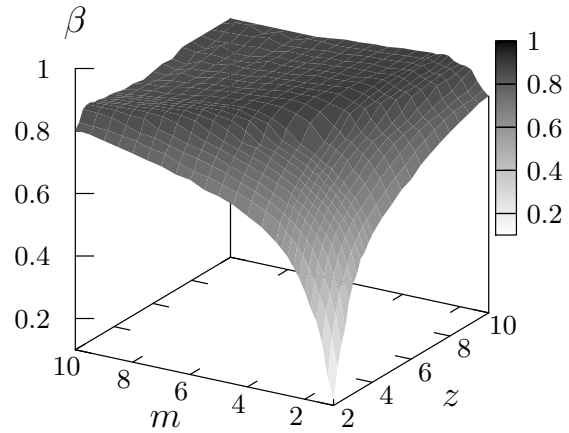


FIG. 4: Critical exponent  $\beta$  as a function of  $z$  and  $m$ .

### 3. Conclusions

We have characterised the scaling behaviour of the Lyapunov exponent at the transition to chaos in a generic map exhibiting type-III intermittency by means of a critical exponent  $\beta$ . We have shown that  $\beta$  depends on both, the order of the tangency of the map, described by the parameter  $z$ , and the probability of reinjection into the laminar region, expressed by the parameter  $m$ . We have calculated numerically the critical exponent on the space of parameters  $(z, m)$  and have obtained that  $\beta$  varies smoothly on the interval  $(0, 1)$ , in contrast with earlier predictions for the scaling behaviour of the Lyapunov exponent in type-III intermittency. It is to be expected that the scaling behaviour shown here for the Lyapunov exponent could also be detected in experimental situations with type-III intermittent chaos, where deviations from the scaling properties of the average laminar length have already been found.

### Acknowledgements

This work was supported by CDCHT, Universidad de Los Andes, Mérida, and by FONACIT (Venezuela) under grants C-1396-06-05-B and F-2002000426, respectively. Calculations were performed at CEMVICC, Universidad de Carabobo, Valencia (Venezuela). M. G. C. thanks the Consortium of The Americas for Interdisciplinary Science at the University of New Mexico, Albuquerque, U.S.A., for hospitality while part of this work was carried out.

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